Clifford Algebra Based Polydimensional Relativity and Relativistic Dynamics[1]

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November, 2000

Starting from the geometric calculus based on Clifford algebra, the idea that physical quantities are Clifford aggregates ("polyvectors") is explored. A generalized point particle action ("polyvector action") is proposed. It is shown that the polyvector action, because of the presence of a *scalar* (more precisely a *pseudoscalar*) variable, can be reduced to the well known, unconstrained, Stueckelberg action which involves an invariant evolution parameter. It is pointed out that, starting from a different direction, DeWitt and Rovelli postulated the existence of a *clock* variable attached to particles which serve as a reference system for identification of spacetime points. The action they postulated is equivalent to the polyvector action. Relativistic dynamics (with an invariant evolution parameter) is thus shown to be based on even stronger theoretical and conceptual foundations than usually believed.

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[1]Talk presented at the IARD 2000 Conference, 26–28 June, 2000.

1 Introduction

In the history of physics it has often happened that a good new formalism contained also good new physics waiting to be discovered and identified in suitable experiments. Today the so called *Fock-Schwinger proper time formalism* is widely recognized for its elegance and usefulness, especially when considering quantum fields in curved spaces. There are two main interpretations of the formalism:

(i) According to the first one, it is considered merely as a useful calculational tool, without a physical significance. Evolution in τ and the absence of the constraint is assumed to be fictitious and unphysical. In order to make contact with physics one has to get rid of τ in all considered expressions by integrating them over τ. By doing so one projects unphysical expressions into the physical ones, and in particular one projects unphysical states into the physical ones.

(ii) According to the second interpretation, evolution in τ is genuine and physical. There is indeed the dynamics in spacetime. Mass is a constant of motion and not a fixed constant in the Lagrangian. Such an approach was proposed by Fock [1] and subsequently investigated by Stueckelberg [2], Feynman [3], Schwinger [4], Davidon [5], Horwitz [6], Fanchi [7] and many others [8,9].

In this paper I am going to show that yet another, widely investigated formalism based on Clifford algebra brings a strong argument in favor of the interpretation (ii). Clifford numbers can be used to represent vectors, multivectors and, in general, polyvectors (which are Clifford aggregates). They form a very useful tool for geometry. The well known equations of physics can be cast into elegant compact forms by using the geometric calculus based on Clifford algebra.

These compact forms suggest a generalization, discussed in the literature by Pezzaglia [10], Castro [11] and also in ref. [12], that every physical quantity is a polyvector. For instance, the momentum polyvector in 4-dimensional spacetime has not only a vector part, but also a scalar, bivector, pseudovector and pseudoscalar part. Similarly for the velocity polyvector. Now we can straightforwardly generalize the conventional constrained action by rewriting it in terms of polyvectors. By doing so, we obtain in the action also a term which corresponds to the scalar or pseudoscalar part of the velocity polyvector. A consequence of such an extra term is that, when confining us for simplicity to polyvectors with the pseudoscalar and the vector part only, the variables corresponding to 4-vector components, can all be taken as independent. After a straightforward procedure in which we omit the extra term in the action, since it turns out to be just a total derivative, we obtain the Stueckelberg unconstrained action! This is certainly a remarkable result.

The original, constrained action is equivalent to the unconstrained action. An analogous procedure can be applied also to the extended objects such as strings, membranes or branes in general.

After describing briefly the essence of geometric calculus based on Clifford algebra I am going to show how relativistic dynamics (which contains the invariant evolution parameter) emerges from the Clifford algebra based reformulation and generalization of the theory of relativity. Briefly I am going to touch also few other relevants subjects.

2   Geometric calculus based on Clifford algebra

I am going to discuss the calculus with *vectors* and *their generalizations*[2]. Geometrically, a vector is an oriented line element.

How to multiply vectors? There are two possibilities:

1. *The inner product*

$$a \cdot b = b \cdot a \tag{1}$$

of vectors a and b. The quantity $a \cdot b$ is a *scalar*.

2. *The outer product*

$$a \wedge b = -b \wedge a \tag{2}$$

which is an oriented element of a plane.

The products 1 and 2 can be considered as the *symmetric* and the *antisymmetric* parts of the *Clifford product*, called also *geometric product*

$$ab = a \cdot b + a \wedge b \tag{3}$$

where

$$a \cdot b \equiv \frac{1}{2}(ab + ba) \tag{4}$$

$$a \wedge b \equiv \frac{1}{2}(ab - ba) \tag{5}$$

This suggests a generalization to trivectors, quadrivectors, etc. It is convenient to introduce the name *r-vector* and call r its *degree*:

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[2]Here I shall provide a brief, simplified introduction into the subject. A more elaborated discussion will be provided elsewhere ([13]).

0-vector	s	scalar
1-vector	a	vector
2-vector	$a \wedge b$	bivector
3-vector	$a \wedge b \wedge c$	trivector
.	.	.
.	.	.
.	.	.
r-vector	$A_r = a_1 \wedge a_2 \wedge ... \wedge a_r$	multivector

In a space of finite dimension this cannot continue indefinitely: the n-vector is the highest r-vector in V_n and the $(n+1)$-vector is identically zero. An r-vector A_r represents an oriented r-volume (or r-direction) in V_n.

Multivectors A_r are the elements of the *Clifford algebra* $\mathcal{C}_n$ of V_n. An element of $\mathcal{C}_n$ will be called a *Clifford number*. Clifford numbers can be multiplied among themselves and the results are Clifford numbers of mixed degrees, as indicated in the basic equation (3). The theory of multivectors, based on Clifford algebra, was developed by Hestenes ([14]). In the following some useful formulas are displayed without proofs.

For a vector a and an r-vector A_r the inner and the outer product are defined according to

$$a \cdot A_r \equiv \frac{1}{2}\left(aA_r - (-1)^r A_r a\right) = -(-1)^r A_r \cdot a \qquad (6)$$

$$a \wedge A_r = \frac{1}{2}\left(aA_r + (-1)^r A_r a\right) = (-1)^r A_r \wedge a \qquad (7)$$

The inner product has symmetry opposite to that of the outer product, therefore the signs in front of the second terms in the above equations are different.

Combining (6) and (7) we find

$$aA_r = a \cdot A_r + a \wedge A_r \qquad (8)$$

For $A_r = a_1 \wedge a_2 \wedge ... \wedge a_r$ eq.(6) can be evaluated to give the useful expansion

$$a \cdot (a_1 \wedge ... \wedge a_r) = \sum_{k=1}^{r}(-1)^{k+1}(a \cdot a_k)a_1 \wedge ...a_{k-1} \wedge a_{k+1} \wedge ...a_r \qquad (9)$$

In particular,

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b \qquad (10)$$

It is very convenient to introduce, besides the basis vectors e_μ, another set of basis vectors e^ν by the condition

$$e_\mu \cdot e^\nu = \delta_\mu{}^\nu \qquad (11)$$

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Each e^μ is a linear combination of e_ν:

$$e^\mu = g^{\mu\nu} e_\nu \tag{12}$$

from which we have

$$g^{\mu\alpha} g_{\alpha\nu} = \delta_\mu{}^\nu \tag{13}$$

and

$$g^{\mu\nu} = e^\mu \cdot e^\nu = \frac{1}{2}(e^\mu e^\nu + e^\nu e^\mu) \tag{14}$$

Let e_1, e_2, ..., e_n be linearly independent vectors, and α, α^i, $\alpha^{i_1 i_2}$, ... scalar coefficients. A generic Clifford number can then be written as

$$A = \alpha + \alpha^i e_i + \frac{1}{2!}\, \alpha^{i_1 i_2}\, e_{i_1} \wedge e_{i_2} + ... \frac{1}{n!}\, \alpha^{i_1 ... i_n} e_{i_1} \wedge ... \wedge e_{i_n} \tag{15}$$

Since it is a superposition of multivectors of all possible grades it will be called *polyvector*.[3] Another name, also often used in the literature, is *Clifford aggregate*. These mathematical objects have far reaching geometrical and physical implications that will be discussed and explored to some extent in the rest of the paper.

2.1 The algebra of spacetime

In spacetime we have 4 linearly independent vectors e_μ, $\mu = 0, 1, 2, 3$. Let us consider *flat* spacetime. It is convenient then to take orthonormal basis vectors γ_μ

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} \tag{16}$$

where $\eta_{\mu\nu}$ is the diagonal metric tensor with signature $(+ - --)$.

The Clifford algebra in V_4 is called the *Dirac algebra*. Writing $\gamma_{\mu\nu} \equiv \gamma_\mu \wedge \gamma_\nu$ for a basis bivector, $\gamma_{\mu\nu\rho} \equiv \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho$ for a basis trivector and $\gamma_{\mu\nu\rho\sigma} \equiv \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma$ for a basis quadrivector we can express an arbitrary number of Dirac algebra as

$$D = \sum_r D_r = d + d^\mu \gamma_\mu + \frac{1}{2!}\, d^{\mu\nu}\gamma_{\mu\nu} + \frac{1}{3!}\, d^{\mu\nu\rho}\gamma_{\mu\nu\rho} + \frac{1}{4!}\, d^{\mu\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma} \tag{17}$$

where d, d^μ, $d^{\mu\nu}$, ... are scalar coefficients.

Let us introduce

$$\gamma_5 \equiv \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \ , \qquad \gamma_5^2 = -1 \tag{18}$$

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[3] Following a suggestion by Pezzaglia ([10]) I call a generic Clifford number *polyvector* and reserve the name *multivector* for an r-vector, since the latter name is already widely used for the corresponding object in the calculus of differential forms.

which is the unit element of 4-dimensional volume and is called *pseudoscalar*. Using the relations

$$\gamma_{\mu\nu\rho\sigma} = \gamma_5 \epsilon_{\mu\nu\rho\sigma} \tag{19}$$

$$\gamma_{\mu\nu\rho} = \gamma_{\mu\nu\rho\sigma} \gamma^\rho \tag{20}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor and introducing the new coefficients

$$S \equiv d , \quad V^\mu \equiv d^\mu , \quad T^{\mu\nu} \equiv \frac{1}{2} d^{\mu\nu}$$

$$C_\sigma \equiv \frac{1}{3!} d^{\mu\nu\rho} \epsilon_{\mu\nu\rho\sigma} , \qquad P \equiv \frac{1}{4!} d^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \tag{21}$$

we can rewrite D of eq.(17) as the sum of scalar, vector, bivector, pseudovector and pseudoscalar part:

$$D = S + V^\mu \gamma_\mu + T^{\mu\nu} \gamma_{\mu\nu} + C^\mu \gamma_5 \gamma_\mu + P \gamma_5 \tag{22}$$

2.2 Polyvector fields

A polyvector may depend on spacetime points. Let $A = A(x)$ be an r-vector field. Then one can define the *gradient operator* according to

$$\partial = \gamma^\mu \partial_\mu \tag{23}$$

where ∂_μ is the usual partial derivative. The gradient operator ∂ can act on any r-vector field. Using (8) we have

$$\partial A = \partial \cdot A + \partial \wedge A \tag{24}$$

Example. Let $A = a = a_\nu \gamma^\nu$ be a 1-vector field. Then

$$\begin{aligned}
\partial a &= \gamma^\mu \partial_\mu (a_\nu \gamma^\nu) = \gamma^\mu \cdot \gamma^\nu \, \partial_\mu a^\nu + \gamma^\mu \wedge \gamma^\nu \partial_\mu a_\nu \\
&= \partial_\mu a^\mu + \frac{1}{2} (\partial_\mu a_\nu - \partial_\nu a_\mu) \gamma^\mu \wedge \gamma^\nu
\end{aligned} \tag{25}$$

The simple expression ∂a thus contains a scalar and bivector part, the former being the usual divergence and the latter the usual curl of a vector field.

Maxwell's equations We shall demonstrate now by a concrete physical example the usefulness of Clifford algebra. Let us consider the electromagnetic field which, in the language of Clifford algebra, is a bivector field F. The source of the field is the electromagnetic current j which is a 1-vector field. Maxwell's equations read

$$\partial F = 4\pi j \tag{26}$$

The grade of the gradient operator ∂ is 1. Therefore we can use the relation (24) and we find that eq.(26) becomes

$$\partial \cdot F + \partial \wedge F = 4\pi j \tag{27}$$

which is equivalent to

$$\partial \cdot F = -4\pi j \tag{28}$$

$$\partial \wedge F = 0 \tag{29}$$

since the first term on the left of eq.(27) is a vector and the second term is a bivector. This results from the general relation (27). It can also be explicitly demonstrated. Expanding

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \tag{30}$$

$$j = j^\mu \gamma_\mu \tag{31}$$

we have

$$\begin{aligned}
\partial \cdot F &= \gamma^\alpha \partial_\alpha \cdot \left(\frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu\right) = \frac{1}{2} \gamma^\alpha \cdot (\gamma_\mu \wedge \gamma_\nu) \partial_\alpha F^{\mu\nu} \\
&= \frac{1}{2} \left((\gamma^\alpha \cdot \gamma_\mu)\gamma_\nu - (\gamma^\alpha \cdot \gamma_\nu)\gamma_\mu\right) \partial_\alpha F^{\mu\nu} = \partial_\mu F^{\mu\nu} \gamma_\nu
\end{aligned} \tag{32}$$

$$\partial \wedge F = \frac{1}{2} \gamma^\alpha \wedge \gamma_\mu \wedge \gamma_\nu \, \partial_\alpha F^{\mu\nu} = \frac{1}{2} \epsilon^\alpha{}_{\mu\nu\rho} \, \partial_\alpha F^{\mu\nu} \gamma_5 \gamma^\rho \tag{33}$$

where we have used (10) and eqs.(19),(20). From the above considerations it then follows that the compact equation (26) is equivalent to the usual tensor form of Maxwell's equations

$$\partial_\nu F^{\mu\nu} = -4\pi j^\mu \tag{34}$$

$$\epsilon^\alpha{}_{\mu\nu\rho} \, \partial_\alpha F^{\mu\nu} = 0 \tag{35}$$

Applying the gradient operator ∂ to the left and to the right side of eq.(26) we have

$$\partial^2 F = \partial j \tag{36}$$

Since $\partial^2 = \partial \cdot \partial + \partial \wedge \partial = \partial \cdot \partial$ is a scalar operator, $\partial^2 F$ is a bivector. The right hand side of eq.(36) gives

$$\partial j = \partial \cdot j + \partial \wedge j \tag{37}$$

Equating the terms of the same grade on the left and the right hand side of eq.(36) we obtain

$$\partial^2 F = \partial \wedge j \tag{38}$$

$$\partial \cdot j = 0 \tag{39}$$

The last equation expresses the conservation of the electromagnetic current.

Motion of a charged particle   In this example we wish to go a step forward. Our aim is not only to describe how a charged particle moves in an electromagnetic field, but also include particle's (classical) spin. Therefore, following Pezzaglia ([10]), we define the *momentum polyvector P* as the *vector momentum p* plus the bivector *spin angular momentum S*

$$P = p + S \tag{40}$$

or in components

$$P = p^\mu \gamma_\mu + \frac{1}{2} S^{\mu\nu} \, \gamma_\mu \wedge \gamma_\nu \tag{41}$$

We also assume that the condition $p_\mu S^{\mu\nu} = 0$ is satisfied. The latter condition ensures the spin to be a simple bivector, which is purely spacelike in the rest frame of the particle. The polyvector equation of motion is

$$\dot{P} \equiv \frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{e}{2m} [P, F] \tag{42}$$

where $[P, F] \equiv PF - FP$. The vector and bivector parts of eq.(42) are

$$\dot{p}^\mu = \frac{e}{m} \, F^\mu{}_\nu p^\nu \tag{43}$$

$$\dot{S}^{\mu\nu} = \frac{e}{2m} (F^\mu{}_\alpha S^{\alpha\nu} - F^\nu{}_\alpha S^{\alpha\mu}) \tag{44}$$

These are just the equation of motion for linear momentum and spin, respectively.

2.3   Physical quantities as polyvectors

The compact equations at the end of the last subsection suggest a generalization that every physical quantity is a polyvector. We shall explore such an assumption and see how far we can get.

In 4-dimensional spacetime *the momentum polyvector* is

$$P = \mu + p^\mu e_\mu + S^{\mu\nu} e_\mu e_\nu + \pi^\mu e_5 e_\mu + m e_5 \tag{45}$$

and *the velocity polyvector* is

$$\dot{X} = \dot{\sigma} + \dot{x}^\mu e_\mu + \dot{\alpha}^{\mu\nu} e_\mu e_\nu + \dot{\xi}^\mu e_5 e_\mu + \dot{s} e_5 \tag{46}$$

where e_μ are four basis vectors satisfying

$$e_\mu \cdot e_\nu = \eta_{\mu\nu} \tag{47}$$

and $e_5 \equiv e_0 e_1 e_2 e_3$ is the pseudoscalar. For the purposes which will become clear later we now use the symbols e_μ, e_5 instead of γ_μ and γ_5.

We associate with each particle the velocity polyvector $\dot{X}$ and its conjugate momentum polyvector P. These quantities are generalizations of the point particle 4-velocity $\dot{x}$ and its conjugate momentum p. Besides a vector part we now include the scalar part $\dot{\sigma}$, the bivector part $\dot{\alpha}^{\mu\nu}e_\mu e_\nu$, the pseudovector part $\dot{\xi}^\mu e_5 e_\mu$ and the pseudoscalar part $\dot{s}e_5$ into the definition of a particle's velocity, and analogously for a particle's momentum. We would like now to derive the equations of motion which will tell us how those quantities depend on the evolution parameter τ. For simplicity we consider a free particle.

Let the action be a straightforward generalization of the first order or phase space action of the usual constrained point particle relativistic theory:

$$I[X, P, \lambda] = \frac{1}{2} \int d\tau \left(P\dot{X} + \dot{X}P - \lambda P^2 \right) \tag{48}$$

where λ is a scalar Lagrange multiplier. Variation of (48) with respect to λ gives the constraint

$$P^2 = 0 \tag{49}$$

Using the definition (45), the last equation becomes[4]

$$P^2 = p^2 - m^2 - \pi^2 + \mu^2 + 2\mu(p^\mu e_\mu + me_5) + \text{ etc.} = 0 \tag{50}$$

After quantization the above constraint becomes

$$\hat{P}^2\Phi = 0 \tag{51}$$

where Φ is a polyvector valued wave function, or briefly, *polyvector wave function* [13].

A particular class of solutions satisfies

$$\hat{P}\Psi = 0 \tag{52}$$

In particular, when the state represented by Ψ has definite values $\mu = 0$, $S^{\mu\nu} = 0$, $\pi^\mu = 0$, then

$$\hat{P}\Psi = (\hat{p}^\mu e_\mu + me_5)\Psi = 0 \tag{53}$$

or

$$(\hat{p}^\mu\gamma_\mu - m)\Psi = 0 \tag{54}$$

where

$$\gamma_\mu \equiv e_5 e_\mu \tag{55}$$

$$\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = e_0 e_1 e_2 e_3 \equiv e_5 \tag{56}$$

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[4]For more details see [13].

We have thus found that the *Dirac equation* (54) is a special case of the equation (52), which in turn is the "square root" of the generalized Klein-Gordon equation (51). The latter equation involves the polyvector wave function Ψ. Amongst various possible polyvector wave functions there are such that satisfy eq.(54), i.e., the Dirac equation. The latter equation describes a spin $\frac{1}{2}$ particle, and Ψ satisfying (54) is a spinor. This obviously means that spinors can be represented as a sort of polyvectors.

We have thus arrived at a very important observation, namely that a generic polyvector contains spinors. A generic polyvector wave function contains bosons and fermions.

To illustrate this let us consider the 3-dimensional space V_3. Basis vectors are σ_1, σ_2, σ_3 and they satisfy the Pauli algebra

$$\sigma_i \cdot \sigma_j \equiv \frac{1}{2}(\sigma_i \sigma_j + \sigma_j \sigma_i) = \delta_{ij} , \qquad i,j = 1,2,3 \tag{57}$$

The unit pseudoscalar

$$\sigma_1 \sigma_2 \sigma_3 \equiv I \tag{58}$$

commutes with all elements of the Pauli algebra and its square is $I^2 = -1$. It behaves as the ordinary imaginary unit i. Therefore, in 3-space, we may identify the imaginary unit i with the unit pseudoscalar I.

An arbitrary polyvector in V_3 can be written in the form[5]

$$\Phi = \alpha^0 + \alpha^i \sigma_i + i\beta^i \sigma_i + i\beta = \Phi^0 + \Phi^i \sigma_i \tag{59}$$

where Φ^0, Φ^i are formally complex numbers.

We can decompose ([14]):

$$\Phi = \Phi \frac{1}{2}(1 + \sigma_3) + \Phi \frac{1}{2}(1 - \sigma_3) = \Phi_+ + \Phi_- \tag{60}$$

where $\Phi \in \mathcal{I}_+$ and $\Phi_- \in \mathcal{I}_-$ are independent minimal *left ideals*.

Let us recall the definition of ideal. A left ideal $\mathcal{I}_L$ in an algebra C is a set of elements such that if $a \in \mathcal{I}_L$ and $c \in C$, then $ca \in \mathcal{I}_L$. If $a \in \mathcal{I}_L$, $b \in \mathcal{I}_L$, then $(a + b) \in \mathcal{I}_L$. A right ideal $\mathcal{I}_R$ is defined similarly except that $ac \in \mathcal{I}_R$. A left (right) minimal ideal is a left (right) ideal which contains no other ideals but itself and the null ideal.

A basis in $\mathcal{I}_+$ is given by two polyvectors

$$u_1 = \frac{1}{2}(1 + \sigma_3) , \quad u_2 = (1 - \sigma_3)\sigma_1 \tag{61}$$

which satisfy

$$
\begin{aligned}
\sigma_3 u_1 &= u_1 & \sigma_1 u_1 &= u_2 & \sigma_2 u_1 &= iu_2 \\
\sigma_3 u_2 &= -u_2 & \sigma_1 u_2 &= u_1 & \sigma_2 u_2 &= -iu_1
\end{aligned}
\tag{62}
$$

[5] Here I review and adapt the Hestenes procedure ([14]).

These are precisely the well known relations for basis spinors. Thus we have arrived at the very profound result that the polyvectors u_1, u_2 behave as basis spinors.

Similarly, a basis in $\mathcal{I}_+$ is given by

$$v_1 = \frac{1}{2}(1 + \sigma_3)\sigma_1 \ , \quad v_2 = \frac{1}{2}(1 - \sigma_3) \tag{63}$$

and satisfies

$$\begin{aligned}
\sigma_3 v_1 &= v_1 & \sigma_1 v_1 &= v_2 & \sigma_2 v_1 &= iv_2 \\
\sigma_3 v_2 &= -v_2 & \sigma_1 v_2 &= v_1 & \sigma_2 v_2 &= -iv_1
\end{aligned} \tag{64}$$

A polyvector Φ can be written in *spinor basis*

$$\Phi = \Phi_+^1 u_1 + \Phi_+^2 u_2 + \Phi_-^1 v_1 + \Phi_-^2 v_2 \tag{65}$$

where

$$\begin{aligned}
\Phi_+^1 &= \Phi^0 + \Phi^3 \ , & \Phi_-^1 &= \Phi^1 - i\Phi^2 \\
\Phi_+^2 &= \Phi^1 + i\Phi^2 \ , & \Phi_-^2 &= \Phi^0 - \Phi^3
\end{aligned} \tag{66}$$

Eq.(65) is an alternative expansion of a polyvector. We can expand the same polyvector Φ either according to (59) or according to (65).

Introducing the matrices

$$\xi_{ab} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \qquad \Phi^{ab} = \begin{pmatrix} \Phi_+^1 & \Phi_-^1 \\ \Phi_+^2 & \Phi_-^2 \end{pmatrix} \tag{67}$$

we can write (65) as

$$\Phi = \Phi^{ab}\xi_{ab} \tag{68}$$

Thus a polyvector can be represented as a *matrix Φ^{ab}*. The decomposition (60) then reads

$$\Phi = \Phi_+ + \Phi_- = (\Phi_+^{ab} + \Phi_-^{ab})\xi_{ab} \tag{69}$$

where

$$\Phi_+^{ab} = \begin{pmatrix} \Phi_+^1 & 0 \\ \Phi_+^2 & 0 \end{pmatrix} \tag{70}$$

$$\Phi_-^{ab} = \begin{pmatrix} 0 & \Phi_-^1 \\ 0 & \Phi_-^2 \end{pmatrix} \tag{71}$$

From (68) we can directly calculate the matrix elements Φ^{ab}. We only need to introduce the new elements $\xi^{\dagger ab}$ which satisfy

$$(\xi^{\dagger ab}\xi_{cd})_S = \delta^a{}_c \delta^b{}_d \tag{72}$$

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The superscript † means Hermitian conjugation $(^{14})$. If

$$A = A_S + A_V + A_B + A_P \tag{73}$$

is a Pauli number, then

$$A^\dagger = A_S + A_V - A_B - A_P \tag{74}$$

This means that the order of basis vectors σ_i in the expansion of $A^\dagger$ is reversed. Thus $u_1^\dagger = u_1$, but $u_2^\dagger = \frac{1}{2}(1 + \sigma_3)\sigma_1$. Since $(u_1^\dagger u_1)_S = \frac{1}{2}$, $(u_2^\dagger u_2)_S = \frac{1}{2}$ it is convenient to introduce $u^{\dagger^1} = 2u_1$ and $u^{\dagger^2} = 2u_2$ so that $(u^{\dagger^1} u_1)_S = 1$, $(u^{\dagger^2} u_2)_S = 1$. If we define similar relations for v_1, v_2 then we obtain (72).

From (68) and (72) we have

$$\Phi^{ab} = (\xi^{\dagger^{ab}} \Phi)_I \tag{75}$$

Here the subscript I means *invariant part*, i.e. scalar plus pseudoscalar part (remember that pseudoscalar unit has here the role of imaginary unit and that Φ^{ab} are thus complex numbers).

The relation (75) tells us how from an arbitrary polyvector Φ (i.e. a Clifford number) can we obtain its *matrix representation* Φ^{ab}.

Φ in (75) is an arbitrary Clifford number. In particular Φ may be any of the basis vectors σ_i.

Example $\Phi = \sigma_1$:

$$
\begin{aligned}
\Phi^{11} &= (\xi^{\dagger^{11}} \sigma_1)_I = (u^{\dagger^1} \sigma_1)_I = ((1 + \sigma_3)\sigma_1)_I = 0 \\
\Phi^{12} &= (\xi^{\dagger^{12}} \sigma_1)_I = (v^{\dagger^1} \sigma_1)_I = ((1 - \sigma_3)\sigma_1 \sigma_1)_I = 1 \\
\Phi^{21} &= (\xi^{\dagger^{21}} \sigma_1)_I = (u^{\dagger^2} \sigma_1)_I = ((1 + \sigma_3)\sigma_1 \sigma_1)_I = 1 \\
\Phi^{22} &= (\xi^{\dagger^{22}} \sigma_1)_I = (v^{\dagger^2} \sigma_1)_I = ((1 - \sigma_3)\sigma_1)_I = 0
\end{aligned}
\tag{76}
$$

Therefore

$$(\sigma_1)^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{77}$$

Similarly we obtain from (75) when $\Phi = \sigma_2$ and $\Phi = \sigma_3$, respectively, that

$$(\sigma_2)^{ab} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad (\sigma_3)^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{78}$$

So we have obtained the matrix representation of the basis vectors σ_i. Actually (77),(78) are the well known *Pauli matrices*.

When $\Phi = u_1$ and $\Phi = u_2$, respectively, we obtain

$$(u_1)^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \qquad (u_2)^{ab} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{79}$$

which are a matrix representation of the *basis spinors* u_1 and u_2.

Similarly we find

$$(v_1)^{ab} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad (v_2)^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{80}$$

In general a *spinor* is a superposition

$$\psi = \psi^1 u_1 + \psi^2 u_2 \tag{81}$$

and its matrix representation is

$$\psi \rightarrow \begin{pmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{pmatrix} \tag{82}$$

Another independent spinor is

$$\chi = \chi^1 v_1 + \chi^2 v_2 \tag{83}$$

with matrix representation

$$\chi \rightarrow \begin{pmatrix} 0 & \chi^1 \\ 0 & \chi^2 \end{pmatrix} \tag{84}$$

If we multiply a spinor ψ from the left by any element R of the Pauli algebra we obtain another spinor

$$\psi' = R\psi \rightarrow \begin{pmatrix} \psi'^1 & 0 \\ \psi'^2 & 0 \end{pmatrix} \tag{85}$$

which is an element of the same minimal left ideal. Therefore, if only multiplication from the left is considered, a spinor can be considered as a column matrix

$$\psi \rightarrow \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \tag{86}$$

This is just the common representation of spinors. But it is not general enough to be valid for all the interesting situations which occur in the Clifford algebra.

We have thus arrived at a very important finding. *Spinors* are just particular *Clifford numbers*: they belong to a left or right minimal ideal. For instance, a generic *spinor* is

$$\psi = \psi^1 u_1 + \psi^2 u_2 \quad \text{with} \quad \Phi^{ab} = \begin{pmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{pmatrix} \tag{87}$$

A *conjugate spinor* is

$$\psi^\dagger = \psi^{1*} u_1^\dagger + \psi^{2*} u_2^\dagger \quad \text{with} \quad (\Phi^{ab})^* = \begin{pmatrix} \psi^{1*} & \psi^{2*} \\ 0 & 0 \end{pmatrix} \tag{88}$$

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and it is an element of a minimal *right ideal*.

The above consideration can be generalized to 4 or more dimensions (see (^{15})).

Scalars, vectors, etc., and spinors can be reshuffled by the elements of Clifford algebra. For instance, vectors can be transformed into spinors, and vice verse. Within Clifford algebra we have thus transformations which change bosons into fermions! It remains to be investigated whether such a kind of "supersymmetry" is related to the well known supersymmetry.

2.4 Relativity of signature

In the previous subsection we have seen how Clifford algebra can be used in the formulation of the point particle classical and quantum theory. The metric of spacetime was assumed as usually to have the Minkowski signature, and we have used the choice $(+ - --)$. We are now going to find out that within Clifford algebra the signature is a matter of choice of basis vectors amongst the available Clifford numbers.

Suppose we have a 4-dimensional space V_4 with signature $(+ + + +)$. Let e_μ, $\mu = 0, 1, 2, 3$ be basis vectors satisfying

$$e_\mu \cdot e_\nu \equiv \frac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = \delta_{\mu\nu} \tag{89}$$

where $\delta_{\mu\nu}$ is the *Euclidean signature* of V_4. The vectors e_μ can be used as generators of Clifford algebra $\mathcal{C}$ over V_4 with a generic Clifford number (named also polyvector or Clifford aggregate) expanded in term of $e_J = (1, e_\mu, e_{\mu\nu}, e_{\mu\nu\alpha}, e_{\mu\nu\alpha,\beta})$, $\mu < \nu < \alpha < \beta$,

$$A = a^J e_J = a + a^\mu e_\mu + a^{\mu\nu} e_\mu e_\nu + a^{\mu\nu\alpha} e_\mu e_\nu e_\alpha + a^{\mu\nu\alpha\beta} e_\mu e_\nu e_\alpha e_\beta \tag{90}$$

Let us consider the set of four Clifford numbers $(e_0, e_i e_0)$, $i = 1, 2, 3$ and denote them as

$$\begin{aligned} e_0 &\equiv \gamma_0 \\ e_i e_0 &\equiv \gamma_i \end{aligned} \tag{91}$$

The Clifford numbers γ_μ, $\mu = 0, 1, 2, 3$ satisfy

$$\frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu} \tag{92}$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the *Minkowski tensor*. We see that the γ_μ behave as basis vectors in a 4-dimensional space $V_{1,3}$ with signature $(+---)$. We can form a Clifford aggregate

$$\alpha = \alpha^\mu \gamma_\mu \tag{93}$$

which has the properties of a *vector* in $V_{1,3}$. From the point of view of the space V_4 the same object α is a linear combination of a vector and bivector:

$$\alpha = \alpha^0 e_0 + \alpha^i e_i e_0 \tag{94}$$

We may use γ_μ as generators of the Clifford algebra $\mathcal{C}_{1,3}$ defined over the pseudo-Euclidean space $V_{1,3}$. The basis elements of $\mathcal{C}_{1,3}$ are $\gamma_J = (1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\alpha}, \gamma_{\mu\nu\alpha\beta})$, with $\mu < \nu < \alpha < \beta$. A generic Clifford aggregate in $\mathcal{C}_{1,3}$ is given by

$$B = b^J \gamma_J = b + b^\mu \gamma_\mu + b^{\mu\nu} \gamma_\mu \gamma_\nu + b^{\mu\nu\alpha} \gamma_\mu \gamma\nu\gamma_\alpha + b^{\mu\nu\alpha\beta} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \tag{95}$$

With suitable choice of the coefficients $b^J = (b, b^\mu, b^{\mu\nu}, b^{\mu\nu\alpha}, b^{\mu\nu\alpha,\beta})$ we have that B of eq.(95) is equal to A of e.(90). Thus the same number A can be described either within $\mathcal{C}_4$ or within $\mathcal{C}_{1,3}$. The expansions (95) and (90) exhaust all possible numbers of the Clifford algebras $\mathcal{C}_{1,3}$ and $\mathcal{C}_4$. The algebra $\mathcal{C}_{1,3}$ is isomorphic to the algebra $\mathcal{C}_4$ and actually they are just two different representations of the same set of Clifford numbers (called also polyvectors or Clifford aggregates).

As an alternative to (91) we can choose

$$\begin{aligned} e_0 e_3 &\equiv \tilde{\gamma}_0 \\ e_i &\equiv \tilde{\gamma}_i \end{aligned} \tag{96}$$

from which we have

$$\frac{1}{2}(\tilde{\gamma}_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \tilde{\gamma}_\mu) = \tilde{\eta}_{\mu\nu} \tag{97}$$

with $\tilde{\eta}_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Obviously $\tilde{\gamma}_\mu$ are basis vectors of a pseudo-Euclidean space $\tilde{V}_{1,3}$ and they generate the Clifford algebra over $\tilde{V}_{1,3}$ which is yet another representation of the same set of objects (i.e. polyvectors). But the spaces V_4, $V_{1,3}$ and $\tilde{V}_{1,3}$ are not the same and they span different subsets of polyvectors. In a similar way we can obtain spaces with signatures $(+-++)$, $(++-+)$, $(+++-)$, $(-+--)$, $(--+-)$, $(---+)$ and corresponding higher dimensional analogs. But we cannot obtain signatures of the type $(++--)$, $(+-+-)$, etc. In order to obtain such signatures we proceed as follows.

4-space. First we observe that the bivector $\bar{I} = e_3 e_4$ satisfies $\bar{I}^2 = -1$, commutes with e_1, e_2 and anticommutes with e_3, e_4. So we obtain that the set of Clifford numbers $\gamma_\mu = (e_1 \bar{I}, e_2 \bar{I}, e_3, e_3)$ satisfies

$$\gamma_\mu \cdot \gamma_\nu = \bar{\eta}_{\mu\nu} \tag{98}$$

where $\bar{\eta} = \mathrm{diag}(-1, -1, 1, 1)$.

8-space.   Let e_A be basis vectors of 8-dimensional[6] vector space with signature $(+ + + + + + + +)$. Let us decompose

$$e_A = (e_\mu, e_{\bar\mu}) \qquad \begin{aligned} \mu &= 0, 1, 2, 3 \\ \bar\mu &= \bar{0}, \bar{1}, \bar{2}, \bar{3} \end{aligned} \tag{99}$$

The inner product of two basis vectors

$$e_A \cdot e_B = \delta_{AB} \tag{100}$$

then splits into the following set of equations:

$$\begin{aligned} e_\mu \cdot e_\nu &= \delta_{\mu\nu} \\ e_{\bar\mu} \cdot e_{\bar\nu} &= \delta_{\bar\mu\bar\nu} \\ e_\mu \cdot e_{\bar\nu} &= 0 \end{aligned} \tag{101}$$

The number $\bar{I} = e_{\bar 0} e_{\bar 1} e_{\bar 2} e_{\bar 3}$ has the propoerties

$$\begin{aligned} \bar{I}^2 &= 1 \\ \bar{I} e_\mu &= e_\mu \bar{I} \\ \bar{I} e_{\bar\mu} &= -e_{\bar\mu} \bar{I} \end{aligned} \tag{102}$$

The set of numbers

$$\begin{aligned} \gamma_\mu &= e_\mu \\ \gamma_{\bar\mu} &= e_{\bar\mu} \bar{I} \end{aligned} \tag{103}$$

satisfies

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= \delta_{\mu\nu} \\ \gamma_{\bar\mu} \cdot \gamma_{\bar\nu} &= -\delta_{\mu\nu} \\ \gamma_\mu \cdot \gamma_{\bar\mu} &= 0 \end{aligned} \tag{104}$$

The numbers $(\gamma_\mu, \gamma_{\bar\mu})$ thus form a set of basis vectors of a vector space $V_{4,4}$ with signature $(+ + + + - - - -)$.

10-space.   Let $e_A = (e_\mu, e_{\bar\mu})$, $\mu = 1, 2, 3, 4, 5$; $\bar\mu = \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}$ be basis vectors of a 10-dimensional Euclidean space V_{10} with signature $(+ + + ...)$. We introduce $\bar{I} = e_{\bar 1} e_{\bar 2} e_{\bar 3} e_{\bar 4} e_{\bar 5}$ which satisfies

$$\begin{aligned} \bar{I}^2 &= 1 \\ e_\mu \bar{I} &= -\bar{I} e_\mu \\ e_{\bar\mu} \bar{I} &= \bar{I} e_{\bar\mu} \end{aligned} \tag{105}$$

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[6] The Clifford Algebra of 8-dimensional space was studied in ref. ([16]), where it was shown that octonions are imbedded in $\mathcal{C}_8$.

Then the Clifford numbers

$$\begin{aligned}
\gamma_\mu &= e_\mu \bar{I} \\
\gamma_{\bar\mu} &= e_\mu
\end{aligned} \tag{106}$$

satisfy

$$\gamma_\mu \cdot \gamma_\nu = -\delta_{\mu\nu}$$
$$\gamma_{\bar\mu} \cdot \gamma_{\bar\nu} = \delta_{\bar\mu\bar\nu}$$
$$\gamma_\mu \cdot \gamma_{\bar\mu} = 0 \tag{107}$$

The set $\gamma_A = (\gamma_\mu, \gamma_{\bar\mu})$ therefore spans the vector space of signature $(----$ $-+++++)$.

The examples above demonstrate how vector spaces of various signatures are obtained within a given set of polyvectors. Namely, vector spaces of different signature are different subsets of polyvectors within the same Clifford algebra.

This has important physical implications. We have argued that physical quantities are polyvectors (Clifford numbers or Clifford aggregates). Physical space is then not simply a vector space (e.g. Minkowski space), but a space of polyvectors. The latter is a pandimensional continuum $\mathcal{P}$ (^{10}) of points, lines, planes, volumes, etc., altogether. Minkowski space is then just a sub-space with pseudo-Euclidean signature. Other sub-spaces with other signatures also exist within the pandimensional continuum $\mathcal{P}$ and they all have physical significance. If we describe a particle as moving in Minkowski spacetime $V_{1,3}$ we consider only certain physical aspects of the considered object. We have omitted its other physical properties like spin, charge, magnetic moment, etc. We can as well describe the same object as moving in an Euclidean space V_4. Again such a description would reflect only a part of the underlying physical situation described by Clifford algebra.

3 The unconstrained action from the polyvector action

Let us consider the polyvector action (48) and the constraint (49). It is a polyvector equation, i.e. the sum of multivector parts of different degrees. Each multivector part has to vanish separately. Denoting the r-vector part as $\langle P^2 \rangle_r$, eq.(49) can be rewritten as a set of equations

$$\langle P^2 \rangle_r = 0 \ , \quad r = 0, 1, 2, 3, 4 \tag{108}$$

After some straightforward algebra we find

$$\pi^\mu = 0 \ , \quad \mu = 0 \ , \quad S^{\mu\nu} = 0 \tag{109}$$

$$p^\mu p_\mu - m^2 = 0 \tag{110}$$

Therefore the polymomentum and the polyvelocity acquire the simplified forms

$$P = p^\mu e_\mu + m e_5 \tag{111}$$

$$\dot{X} = \dot{x}^\mu + \dot{s} e_5 \tag{112}$$

and the action (48) simplifies to the following phase space action

$$I[s, m, x^\mu, p_\mu, \lambda] = \int d\tau \left[-m\dot{s} + p_\mu \dot{x}^\mu - \frac{\lambda}{2}(p^\mu p_\mu - m^2) \right] \tag{113}$$

which, besides (x^μ, p_μ), has the additional variables (s, m).

The equations of motion resulting from (113) are

$$\delta s \quad : \quad \dot{m} = 0 \tag{114}$$

$$\delta m \quad : \quad \dot{s} - \lambda m = 0 \tag{115}$$

$$\delta x^\mu \quad : \quad \dot{p}_\mu = 0 \tag{116}$$

$$\delta p_\mu \quad : \quad \dot{x}^\mu - \lambda p^\mu = 0 \tag{117}$$

$$\delta \lambda \quad : \quad p^\mu p_\mu - m^2 = 0 \tag{118}$$

We see that in this dynamical system the mass m is one of the dynamical variables; it is canonically conjugate to the variable s. From the equations of motion we easily read out that s is the proper time. Namely, from (115),(117) and (118) we have

$$p^\mu = \frac{\dot{x}^\mu}{\lambda} = m\frac{dx^\mu}{ds} \tag{119}$$

$$\dot{s}^2 = \lambda^2 m^2 = \dot{x}^2 \quad , \text{i.e} \quad ds^2 = dx^\mu dx_\mu \tag{120}$$

Using eq.(115) we find that

$$- m\dot{s} + \frac{\lambda}{2}m^2 = -\frac{m\dot{s}}{2} = -\frac{1}{2}\frac{d(ms)}{d\tau} \tag{121}$$

The action (113) then becomes

$$I = \int d\tau \left(\frac{1}{2}\frac{d(ms)}{d\tau} + p_\mu \dot{x}^\mu - \frac{\lambda}{2}p^\mu p_\mu \right) \tag{122}$$

where λ should be no longer considered as a quantity to be varied, but it is now fixed: $\lambda = \Lambda(\tau)$. The total derivative in (122) can be omitted, and the action is simply

$$I[x^\mu, p_\mu] = \int d\tau (p_\mu \dot{x}^\mu - \frac{\Lambda}{2}p^\mu p_\mu) \tag{123}$$

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For a Λ which is independent of τ, (123) is just *the Stueckelberg action* ([2]). The equations of motion derived from (123) are

$$\dot{x}^\mu - \Lambda p^\mu = 0 \tag{124}$$

$$\dot{p}_\mu = 0 \tag{125}$$

From (125) it follows that $p_\mu p^\mu$ is a constant of motion. Denoting the latter constant of motion as m and using (124) we obtain that momentum can be written as

$$p^\mu = m \frac{\dot{x}^\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = m \frac{\mathrm{d}x^\mu}{\mathrm{d}s} , \quad \mathrm{d}s = (\mathrm{d}x^\mu \mathrm{d}x_\mu)^{1/2} \tag{126}$$

which is the same as in eq.(119). The equations of motion for x^μ and p_μ derived from the Stueckelberg action (123) are the same as the equations of motion derived from the action (113). *A generic Clifford algebra action (48) thus leads directly to the Stueckelberg action.*

The above analysis can be easily repeated for a more general case, by introducing a scalar constant κ^2, so that instead of (113) we have

$$I[s, m, x^\mu, p_\mu, \lambda] = \int \mathrm{d}\tau \left[-m\dot{s} + p_\mu \dot{x}^\mu - \frac{\lambda}{2}(p^\mu p_\mu - m^2 - \kappa^2) \right] \tag{127}$$

Then, instead of (123), we obtain

$$I[x^\mu, p_\mu] = \int \mathrm{d}\tau \left(p_\mu \dot{x}^\mu - \frac{\Lambda}{2}(p^\mu p_\mu - \kappa^2) \right) \tag{128}$$

The corresponding Hamiltonian is

$$H = \frac{\Lambda}{2}(p^\mu p_\mu - \kappa^2) \tag{129}$$

and in the quantized theory the Schrödinger equation reads

$$i\frac{\partial \psi}{\partial \tau} = \frac{\Lambda}{2}(p^\mu p_\mu - \kappa^2)\psi \tag{130}$$

If we derive from (127) the equations of motion (which are straightforward generalizations, for $\kappa \neq 0$, of eqs.(114)-(118)), and eliminate the conjugate variables p_μ and m, we can re-express the action (127) as

$$I[x^\mu, s] = \kappa \int \mathrm{d}\tau (\dot{x}^\mu \dot{x}_\mu - \dot{s}^2)^{1/2} \tag{131}$$

This is a straightforward generalization of the usual relativistic point particle action to an extra variable $s(\tau)$. It is important to bear in mind that this extra variable s is not due to a postulated extra dimension of spacetime, but

due to the existence of the Clifford algebra generated by the basis vectors of spacetime. Although spacetime remains 4-dimensional, a point particle is described not only by four coordinates variables $x^\mu(\tau)$, but also by an extra variable $s(\tau)$.

The extra variable s has brought us to what appears (in the specific case considered) as an $O(1,4)$ invariant action. The "$O(1,4)$" action contains the constraint, therefore the extra variable is not a variable at all (at least if we choose the remaining ones – i.e., x^μ – as the true variables). The extra variable s is related to the parameter τ through a choice of "gauge", that is by choice of the Lagrange multiplier λ. In the particular case we first chose $\lambda = \Lambda(\tau)$. Further we have chosen $\Lambda(\tau)$ as independent of τ. Then one finds $s = \Lambda m \tau$. In such a choice of parametrization s is proportional to τ. Other parametrizations are, of course, possible, and in this respect the "$O(1,4)$" action goes beyond Stueckelberg. But physically it is equivalent to the Stueckelberg action, because an arbitrary choice of gauge (parametrization) has no influence on physics. In short, if we choose x^μ as the dynamical variables (evolution in spacetime, relativistic dynamics), then the s is not a variable at all[7]; it can be chosen to be equal, or at least proportional to τ. And most important, the "$O(1,4)$" action does not come from a space $V_{1,4}$, but from the Clifford algebra over $V_{1,3}$.

4 The polyvector action and DeWitt–Rovelli material reference fluid

In a remarkable paper ([17]) Rovelli considered in modern language the famous Einstein "hole argument" which shows that points of empty spacetime cannot be identified. For a precise formulation the reader is adviced to have a look at Rovelli's paper. Here I present the argument, as I understand it, in a simplified and compact way.

We are familiar with the fact that the Einstein equations are invariant under general coordinates transformations. In a given coordinate system O, let $g_{\mu\nu}(x)$, $X_i^\mu(\tau)$ be a solution to the Einstein equations - a possible universe U, with the metric $g_{\mu\nu}(x)$ and the set of point particles' world lines $X_i^\mu(\tau)$, $i = 1, 2, ..., N$. The same universe U can be expressed in a different coordinate system O' as $g'_{\mu\nu}(x')$, $X_i'^\mu(\tau')$ which, of course, is also a solution to the Einstein equations. This transformation is called also a *passive diffeomorphism*.

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[7]Analogously, in the usual theory of relativity, because of the mass shell constraint, $x^0 \equiv t$ is not a variable at all, and it is often considered as the evolution parameter: so one obtains the evolution in 3-space, but not in spacetime $V_{1,3}$.

Let us now consider another kind of transformation, namely an *active diffeomorphism* which, in the same coordinate system O sends a universe U, described by $g_{\mu\nu}(x)$, $X_i^\mu(\tau)$ into another universe U', described by $g'_{\mu\nu}(x)$, $X_i'^\mu(\tau)$. There is a lot of freedom in choosing active diffeomorphisms. Can then the universes U and U' be physically distinct?

The same initial conditions should lead to the same physical universes. But active diffeomorphisms allow for the possibility that, starting from the same initial conditions at a given spacelike hypersurface (where U and U' are identical), we can arrive at the situation where U and U' are distinct at a "later" spacelike hypersurface. If U and U' were *physically* distinct, then determinism would be violated. Hence U and U' must be physically the same (even if described by different sets of variables related by an active diffeomorphism). But, being the same, spacetime points in the *holes* within matter configuration (the latter being described by the set of worldlines $X_i^\mu(\tau)$) cannot be identified.

If we wish to build up a theory in which spacetime points could be identified, we have to fill spacetime with a *reference fluid*. Such an idea was earlier considered by DeWitt ([18]), and revived by Rovelli ([17]). As a starting point Rovelli considers a simplified *reference system* consisting of a *single particle* and a *clock* attached to it. Variables are then particle's coordinates $X^\mu(\tau)$ and *the clock variable $T(\tau)$*. As a model of general relativity + material reference system theory Rovelli considers the action whose matter part is

$$I = m \int \mathrm{d}\tau \left(\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X_\mu}{\mathrm{d}\tau} - \frac{1}{\omega^2} \left(\frac{\mathrm{d}T}{\mathrm{d}\tau} \right)^2 \right)^{1/2} \tag{132}$$

If we make replacement $m \to \kappa$, $T/\omega \to s$, we obtain precisely the action (131) derived from the polyvector action. Our polyvector action can be generalized ([13]) to strings and higher dimensional membranes (*p*-branes). We obtain the *unconstrained action* starting from the constrained action which includes the *pseudoscalar field*. The latter field is a necessary ingredient of the polyvector generalization of the theory. On the other hand, DeWitt and Rovelli have taken a fluid of reference particles and obtained a similar action which involves a *field of the clock variable*.

5 Conclusion

Clifford algebra is an immensely useful language for geometry and physics. It contains quaternions and differential forms as special cases. Equations of physics acquire remarkably condensed forms. There is a lot of room for new physics. It illuminates the role of spinors: they are a special kind of polyvectors. Clifford algebra, together with the conception of physical quantities

as polyvectors (Clifford aggregates), is very likely the language of a future unified theory. What I was able to present here is only a tip of an iceberg[8]

Geometric calculus based on Clifford algebra leads to the point particle action with an extra variable –the clock variable– which enables evolution in spacetime. In other words, our model with the polyvector action allows for the *dynamics* in spacetime (relativistic dynamics). Relativistic dynamics is a necessary consequence of the existence of Clifford algebra as a general tool for description of geometry of spacetime. Moreover, when considering *dynamics of spacetime itself*, such model, in my opinion, provides a natural resolution of "the problem of time" in quantum gravity. A number of researchers have come close to the viewpoint that even in gravity one has to introduce an extra, invariant, parameter which serves the role of evolution time [19,20,21]. The latter parameter, as already stated, in the polyvector generalization of physics is not postulated but is present automatically.

Ackonwledgement

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[8]A slightly greater part of the iceberg is uncovered in ref. [13].

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